STA4513: Statistical Models of Networks	Fall 2014
Lecture 6—October 15, 2014	
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1 Overview

In the last lecture, we introduced the concept of graphing and review its associated measures for studying the limits of bounded degree graphs. In this lecture, we continue this topic by discussing the notion of local convergence in bounded degree graphs and its properties. The materials here are heavily borrowed from Large Networks and Graph Limits by L. Lovász, 2012.

2 Local Convergence in Bounded Degree Graphs

Before defining the notion of convergence, we introduce some notions of distance for measuring the similarity of two bounded degree graphs F and F'. Let $\rho_{F,r}$ be a distribution over graph F which (1) sample random vertex v and (2) return r-neighborhood H of v.

Definition 1 (Sampling Distance of Depth r)**.**

$$\delta^r_{\odot}(F, F') = \left\| \rho_{F,r} - \rho_{F',r} \right\|_{TV},$$

where $\|\mu - \nu\|_{TV} = \sup_A |\mu(A) - \nu(A)|$.

Definition 2 (Sampling Distance)**.**

$$\delta_{\odot}(F, F') = \sum_{r=0}^{\infty} \frac{1}{2^r} \delta^r_{\odot}(F, F').$$

Definition 3 (Local Convergence Sequence)**.** *A sequence of graphs G_n with $|V(G_n)| \to \infty$ is locally convergent if the r-neighborhood densities converges (i.e., $\rho_{G_n,r}\{H\} \to \sigma_r\{H\}$) for every r and r-ball H.*

Theorem 4. *A graph sequence (G_n) is locally convergent if and only if it is Cauchy sequence in the distance metric space $\delta_{\odot}$.*

3 Consistency of Neighborhood Distributions

Neighborhood sampling results in the probability distribution $\rho_{G_n,r}$ for every G_n and $r \in \mathbb{N}$. By local convergence, this distribution converges to the limit distribution σ_r for every fixed $r \in \mathbb{N}^+$. The sequence of these limit distributions $(\sigma_r)_{r\in\mathbb{N}}$ has two special *consistency* properties.

1. σ_r *projectivity*—If we select a random r-ball rooted at u from σ_r (i.e., $(H, u) \sim \sigma_r$) and extract the $(r-1)$-ball H' centered at u, then this (r-1)-ball H' comes from σ_{r-1} (i.e., $(H', u) \sim \sigma_{r-1}$).

2. *Finite version of involution invariant*—Note that a r-ball rooted at u contains some $(r-1)$-balls centered at the neighbors of u. Let $\sigma_r^*(F)$ be the distribution biased by the degree of each root:

$$\sigma_r^*(F) = \frac{deg(F)\sigma_r(F)}{\sum_H deg(H)\sigma_r(H)}.$$

   Select a random r-ball H from σ_r^* (i.e., $(H,u) \sim \sigma_r^*$). Select a random edge uv from the root u. Create two $(r-1)$-balls H' and H'' rooted at u and v alternatively—by deleting the nodes with the distance of more than $r-1$ from a root. If get the same distributions for neighboring $(r-1)$-balls (i.e., $(H',u,v) \overset{d}{=} (H'',v,u)$) and this holds for every $r \geq 1$, the sequence $(\sigma_r)_{r\geq 1}$ is *involution invariant.*

4 Bijective Correspondence

There is a bijective correspondence between consistent involution invariant sequence $(\sigma_r)_{r\geq 1}$ and involution invariant distribution σ on $G^\bullet$. From any probability distribution σ, one can recover probability distribution σ_r on B_r by selecting a random graph and and taking r-ball around its root (i.e., $\sigma \to (\sigma_1, \sigma_2, \dots)$). Conversely, we can obtain σ from every consistent sequence $(\sigma_r)_{r\geq 1}$ by simply defining $\sigma = \sigma_r(F)$ for every r-ball F; i.e., $(\sigma_1, \sigma_2, \dots) \to \sigma$.

Hence, every locally convergent sequence (G_n) gives rise to an involution-invariant distribution σ. This usually refers as *Benjamini-Schramm limit* or *local limit* of the graph sequence. However, unfortunately, there is no easy way to construct a sequence of finite graphs that converges to a given involution invariant (or graphing).

5 Locally Convergent Sequences and Graphing

Conjecture 5 (Aldous-Lyon)**.** *Every involution invariant probability distribution on* $(G^\bullet, \mathcal{A})$ *is the limit of a locally convergent bounded degree graph sequence.*

Theorem 6. *Every involution invariant probability distribution* σ *on* $G^\bullet$ *can be represented by a graphing.*

Example (Grid): Consider the random graph concentrated on infinite planar grid with any possible root on the grid. The graphing for this bounded degree random graph can be constructed as follows. Take two irrational real α and β (independent over irrationals). Then connect every $x \in [0,1)$ to $x+\alpha \ (mod\ 1)$, $x-\alpha \ (mod\ 1)$, $x+\beta \ (mod\ 1)$, and $x-\beta \ (mod\ 1)$.

Recall the graph of graphs $\mathbb{H}$ where $V(\mathbb{H}) = G^\bullet$. For every $(H,u) \in G^\bullet$ and edge $uv \in E(H)$, there is an edge $(H,u)(H,v)$ in $\mathbb{H}$. Let σ be involution invariant probability distribution.

Lemma 7. *If almost all graphs from* σ *have no automorphisms, then*

1. $(\mathbb{H}, \sigma)$ *is a graphing.*
2. $(\mathbb{H}, \sigma)$ *represents* σ*.*

Proof. The proof sketch follows.

Part (1): Select a random rooted graph (H,u) in $G^\bullet$ from σ^* and a random neighbour v of u. By automorphism assumption, (H,u) and (H,v) are (almost) distinct. So $(H,u)(H,v)$ is a uniformly random edge in $\mathbb{H}$. By involution invariant of σ, $(\mathbb{H},(H,u)(H,v)) \stackrel{d}{=} (\mathbb{H},(H,v)(H,u))$. Call this η. So, $\sigma^*(A) = \eta(A \times G^\bullet) = \eta(G^\bullet \times A)$. This implies that $(\mathbb{H},\sigma)$ is measure preserving.

Part (2): Let $(H,v) \sim \sigma$. Connected component of $\mathbb{H}$ containing (H,v) is isomorphic to (H,v). Note that $u \stackrel{\psi}{\to} (H,u)$ leads to different graphs and so ψ is an injective if $V(H) \to G^\bullet$. From definition d adjacency in $\mathbb{H}$, the embedding ψ preserves adjacency and non-adjacency, and the range is a connected component. □

6 Automorphisms and Colored Graphs

Question: How to deal with σ's that have automorphisms?

Answer: Break the symmetries by a random coloring. Define G^+ be the set of triples (H,v,α) where, $(H,v) \in G^\bullet$ and $\alpha : V(H) \to [0,1]$ is a coloring. Define "a graph of weighted graphs" $\mathbb{H}^+$ on G^+. Connect (G,α) to (G',α') if there is an isomorphism from G to G' such that:

1. $\alpha'(i(u)) = \alpha(u)$ for all $u \in V(G)$.
2. $root(G)$ is a neighbor of $root(G')$.

Given a probability distribution σ on $G^\bullet$, we can define a probability distribution σ^+ on G^+ as follows:

1. Select H in $G^\bullet$ from σ (i.e., $(H,u) \sim \sigma$).
2. Assign independent colors from $[0,1]$ to the nodes $V(H)$.

Lemma 8. *If σ is an involution invariant probability distribution on $G^\bullet$, then $B_\sigma \equiv (G^+,\sigma^+)$ is a graphing and it represents σ.*