This lecture closely follows L. Lovasz (2012) and online notes by M. Racz (2010).

1 Overview

In this lecture, we will discuss graph isomorphism and homomorphism, convergence in terms of homomorphism densities, and then define the limit object of a sequence of graphs, which is known as a graphon. We will conclude by defining a metric over the space of graphons which corresponds with the convergence of homomorphism densities.

2 Graphs

We begin by defining simple graphs.

Definition: Simple graphs are graphs with no self loops and no multiple edges. \( G = (V(G), E(G)) \), where \( V(G) \) is the set of vertices and \( E(G) \) is the set of edges. \( E(G) \subseteq V(G) \times V(G) \).

Definition: Two simple graphs \( G \) and \( G' \) are isomorphic when there exists a bijection \( \varphi : V(G) \rightarrow V(G') \) such that \( (i,j) \in E(G) \iff \varphi(i,j) \in E(G') \).

Note: Isomorphism are adjacency-preserving and non-adjacency preserving maps.

Example: In the example, \( G_1 \) is isomorphic to \( G_2 \) but \( G_1 \) is not isomorphic to \( G_3 \).
2.1 Graph Parameters

**Definition:** A graph parameter is any function $f$ defined on all simple graphs that is invariant under isomorphism. In other words, $f(G) = f(\varphi(G)) = (f \circ \varphi)(G)$ for all isomorphism $\varphi$.

**Definition:** Let $F$ and $G$ be simple graphs. A homomorphism from $F$ to $G$ is a map $\varphi : V(F) \to V(G)$ such that $(i, j) \in E(F) \implies (\varphi(i), \varphi(j)) \in E(G)$.

**Note:** We will write $F \to G$ if there exists a homomorphism from $F$ to $G$.

![Figure 2: $G_4$, $G_5$, $G_6$, and $G_3$](image)

**Example:** In the example, $G_5 \to G_4$ because we can map two distinct vertices in $G_5$ to a same vertex in $G_4$. By similar reason, $G_6 \to G_5$. However, there does not exist a homomorphism $G_7 \to G_6$ because of the triangle clique.

**Definition:** $K_n \equiv$ complete simple graph on $[n]$.

**Note:** $K_n \to G$ means $G$ contains a $K$-clique. $G \to K_n$ means $G$ is n-colourable.

**Definition:** $\text{hom}(F,G) \equiv$ number of homomorphisms from $F$ to $G$.

**Definition:** Homomorphism density of $F$ in $G$:

$$t(F,G) = \frac{\text{hom}(F,G)}{|V(G)|^{|V(F)|}} = \frac{\text{hom}(F,G)}{|V(G)|^{|V(F)|}}$$

**Note:** Homomorphism density can be viewed as

$$\mathbb{P}(\text{uniformly random } \varphi \text{ is a homomorphism } F \to G)$$

For a fixed $G$, $F \to \text{hom}(F,G)$, $F \to t(F,G)$, $G \to \text{hom}(F,G)$, $G \to t(F,G)$ are graph parameters.
3 Convergence Sequence of Graphs

**Definition:** A sequence of graphs $G_n$ is convergent when, for every simple graph $F$, $t(F, G_n)$ is convergent. When the limit exists, we will write $t(F) = \lim_{n \to \infty} t(F, G_n)$.

**Example:**

\[
t(K_1) = \lim_{n \to \infty} t(K_1, G_n) = 1
\]

\[
\frac{1}{2} t(K_2) = \frac{1}{2 \text{hom}(K_2, G_n)} = \frac{|E(G_n)|}{|V(G_n)|^2}
\]

**Note:** If $|E(G_n)| \in o(|V(G_n)|^2)$ then $t(K_2) = 0 \Rightarrow t(F) = 0$ for all simple graph $F$ such that $K_2 \to F$.

3.1 Why Homomorphism and $t$?

**Definition:** $\text{inj}(F, G) \equiv$ number of injective homomorphisms.

**Definition:** $\text{ind}(F, G) \equiv$ number of embeddings as induced subgraphs.

If $|V(G)| = n$ and $|V(F)| = k \leq n$ then,

\[
t_{\text{inj}}(F, G) = \frac{\text{inj}(F, G)}{n(n-1) \ldots (n-k+1)}
\]

\[
t_{\text{ind}}(F, G) = \frac{\text{ind}(F, G)}{n(n-1) \ldots (n-k+1)}
\]

(Similar to sampling without replacement).

**Proposition:** As number of vertices grows, the probability of any homomorphism is injective approaches 1. The following bound captures that fact:

\[
|t(F, G) - t_{\text{inj}}(F, G)| \leq \frac{1}{|V(G)|} \left( \frac{|V(F)|}{2} \right) \frac{|V(G)| \to \infty}{|V(G)|} \to 0
\]

We also have relations between number of injective homomorphisms and number of embedded subgraphs:

\[
t_{\text{inj}}(F, G) = \sum_{F' \supseteq F} t_{\text{ind}}(F', G), \quad t_{\text{ind}}(F, G) = \sum_{F' \supseteq F} (-1)^{|E(F')| - |E(F)|} t_{\text{inj}}(F', G)
\]

Note $F'$ ranges over those graphs on the same vertex set as $F$ and including at least those edges present in $F$. 
3.2 Limit Object

If \( \{G_n\} \) is convergent, then what is the limit? Say \( V(G_n) = [n] \). One might guess that \( \lim G_n \) is a graph on \( \mathbb{N}^+ = \{1, 2, \ldots\} \). But this won’t work in the standard model. (Why? It is related to the fact that there is no uniform distribution on \( \mathbb{N} \).)

Limit object is a symmetric measurable function \( W : [0,1]^2 \to [0,1] \) called a graphon. Here, symmetric means \( W(x, y) = W(y, x) \).

For every simple graph \( F \),
\[
t(F, W) = \int_{[0,1]} \prod_{(i,j) \in E(F)} W(x_i, x_j) \, dx_1, \ldots, dx_k = \mathbb{E} \left[ \prod_{(i,j) \in E(F)} W(X_i, X_j) \right]
\]
where the \( X_i \) are i.i.d \( U(0,1) \) random variables.

![Figure 3: W and G](image)

**Example:** In this example, \( t(F, W) = t(F, G) \) We usually call \( W \) of black and white a “pixel picture” or “empirical graphon”.

**Theorem:** For every convergent graph sequence \( G_n \), there is a graphon \( W \) such that \( t(F, G_n) \to t(F, W) \) (for every simple graph \( F \)). Moreover, the function \( W \) is unique up to a measure preserving transformation (m.p.t.).

**Note:** A function \( T \) is said to be a measure-preserving transformation if and only if \( T(U) \equiv U \), where \( U \sim U(0,1) \).

**Definition:** \( W^T \) is the graphon given by \( W^T(x, y) \equiv W(T(x), T(y)) \).

To understand the theorem statement, note that, if \( W \) is a limit, then so is \( W^T \), for any measure-preserving transformation \( T \). Two graphons \( W, W' \) are said to be weakly isomorphic if there are two measure-preserving transformations \( T, T' \) such that \( W^T = W'^T \) almost everywhere.
**Theorem:** Every graphon $W$ arises as the limit of a convergent graph sequence.

\[
\{\text{space of graphons}\}/\{\text{weak isomorphism}\} \\
\cong \\
\{\text{homomorphism densities arising from limits of finite graphs}\} \\
\cong \\
\{\text{convergent graph sequences}\}/\{\text{same limit}\}
\]

**4 Distance Measure of Graphons**

**Definition:** The edit distance is defined as follows:

\[
d_1(W, W') \equiv \|W - W'\|_1 = \int |W(x, y) - W'(x, y)| \, dx \, dy
\]

**Definition:** The cut distance is defined as follows:

\[
d_{\square}(W, W') \equiv \|W - W'\|_{\square} = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} W(x, y) - W'(x, y) \, dx \, dy \right|
\]

**Definition:** The cut-metric is defined as follows:

\[
\delta_{\square}(W, W') \equiv \inf_{\varphi \in \text{m.p.t.}} d_{\square}(W, W'\varphi)
\]

![Figure 4: W1 and W2](image)

**Example:** In the figure above, $d_1(W_1, W_2) = 4/9$, $d_{\square}(W_1, W_2) = 2/9$, $\delta_{\square}(W_1, W_2) = 0$.

**Proposition:** $\|W\|_{\square} \leq \|W\|_1$
Lemma: (weak-regularity lemma): For every graphon $W$, there is a “step function graphon $U$” (see figure) with $k$ parts such that

\[ \| W - U \|_\square < \frac{2}{\sqrt{\log k}}. \]

Figure 5: Partition of a Graphon

Note the partition of $[0, 1]$ needs not to be an equipartition. This lemma provides us with a tool to approximate graphons using step functions.